

# Cycles of linear and semilinear mappings

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## Abstract

We give a canonical form of matrices of a cycle of linear or semilinear mapping  $V_1 \rightarrow V_2 \rightarrow \dots \rightarrow V_t \rightarrow V_1$  in which all  $V_i$  are complex vector spaces, each line is an arrow  $\rightarrow$  or  $\leftarrow$ , and each arrow denotes a linear or semilinear mapping.

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## 1. Introduction

A mapping  $\mathcal{A}$  from a complex vector space  $U$  to a complex vector space  $V$  is *semilinear* if

$$\mathcal{A}(u + u') = \mathcal{A}u + \mathcal{A}u', \quad \mathcal{A}(\alpha u) = \bar{\alpha} \mathcal{A}u$$

for all  $u, u' \in U$  and  $\alpha \in \mathbb{C}$ . We write  $\mathcal{A} : U \rightarrow V$  if  $\mathcal{A}$  is a linear mapping and  $\mathcal{A} : U \dashrightarrow V$  (using a dashed arrow) if  $\mathcal{A}$  is a semilinear mapping.

We give a canonical form of matrices of a *cycle* of linear and semilinear mappings

$$\mathcal{A} : \quad V_1 \xrightarrow{\mathcal{A}_1} V_2 \xrightarrow{\mathcal{A}_2} \dots \xrightarrow{\mathcal{A}_{t-2}} V_{t-1} \xrightarrow{\mathcal{A}_{t-1}} V_t \xrightarrow{\mathcal{A}_t} V_1 \quad (1)$$

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in which each line is a full arrow  $\longrightarrow$  or  $\longleftarrow$ , or a dashed arrow  $\dashrightarrow$  or  $\dashleftarrow$ . Its special cases are the canonical forms of

- matrix pencils, contragredient matrix pencils, and pairs consisting of a linear mapping and a semilinear mapping (i.e., cycles  $V_1 \rightrightarrows V_2$ ,  $V_1 \rightleftarrows V_2$ , and  $V_1 \dashrightarrow V_2$ ); their canonical matrices were obtained by Kronecker [12], Dobrovol'skaya and Ponomarev [3] (see also [10]), and Djoković [2];
- cycles of linear mappings (all arrows in (1) are full); their canonical form is well known in the theory of quiver representations; see [6, Section 11.1] and [14].

The proof is based on the canonical forms of linear and semilinear operators (i.e., cycles  $V_1 \curvearrowright$  and  $V_1 \curvearrowleft$ ) given by Jordan [11, p. 125] and Haantjes [7] (see also Theorem 3). We use the methods of [13].

In Section 2 we formulate Theorem 1 about a canonical form of matrices of (1); in Section 3 we prove it as follows. In Section 3.1 we reduce the proof of Theorem 1 to cycles (1) in which  $t \geq 2$  and the mappings  $\mathcal{A}_1, \dots, \mathcal{A}_{t-1}$  are linear. In Section 3.2 we give a canonical form of the matrices of  $\mathcal{A}_1, \dots, \mathcal{A}_{t-1}$ . In Section 3.3 we give a canonical form of the matrix of  $\mathcal{A}_t$  with respect to those transformations that preserve the matrices of  $\mathcal{A}_1, \dots, \mathcal{A}_{t-1}$ . In Section 3.4 we complete the proof of Theorem 1.

All matrices and vector spaces that we consider are over the field of complex numbers.

## 2. A canonical form of matrices of a cycle

We denote by  $[v]_e$  the coordinate vector of  $v$  in a basis  $e_1, \dots, e_n$ , and by  $S_{e \rightarrow e'}$  the transition matrix from  $e_1, \dots, e_n$  to a basis  $e'_1, \dots, e'_n$ . We write  $\bar{A} := [\bar{a}_{ij}]$  for a matrix  $A = [a_{ij}]$ .

Let  $\mathcal{A} : U \dashrightarrow V$  be a semilinear mapping. We say that an  $m \times n$  matrix  $\mathcal{A}_{fe}$  is the *matrix of  $\mathcal{A}$  in bases  $e_1, \dots, e_n$  of  $U$  and  $f_1, \dots, f_m$  of  $V$*  if

$$[\mathcal{A}u]_f = \overline{\mathcal{A}_{fe}[u]_e} \quad \text{for all } u \in U. \quad (2)$$

Therefore, the columns of  $\mathcal{A}_{fe}$  are  $\overline{[\mathcal{A}e_1]_f}, \dots, \overline{[\mathcal{A}e_n]_f}$ . We write  $\mathcal{A}_e$  instead of  $\mathcal{A}_{ee}$  if  $U = V$ .

The *direct sum* of matrix sequences  $A = (A_1, \dots, A_t)$  and  $B = (B_1, \dots, B_t)$  is the sequence

$$A \oplus B := \left( \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \dots, \begin{bmatrix} A_t & 0 \\ 0 & B_t \end{bmatrix} \right).$$

For each  $k = 1, 2, \dots$ , define the matrices

$$J_k(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \quad (k\text{-by-}k, \lambda \in \mathbb{C}),$$

$$H_{2k}(\mu) := \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix} \quad (2k\text{-by-}2k, \mu \in \mathbb{C}),$$

and

$$F_n = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}, \quad G_n = \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} \quad ((n-1)\text{-by-}n). \quad (3)$$

The following theorem is the main result of the article.

**Theorem 1.** *For each system of linear and semilinear mappings (1), there exist bases of the spaces  $V_1, \dots, V_t$  in which the sequence of matrices of  $\mathcal{A}_1, \dots, \mathcal{A}_t$  is a direct sum, determined by (1) uniquely up to permutation of summands, of sequences of the following form (in which the points denote sequences of identity matrices or  $0_{00}$ ):*

- (i)
  - $(J_n(\lambda), \dots)$  in which  $\lambda \neq 0$ , if the number of dashed arrows in (1) is even,
  - $(J_n(\lambda), \dots)$  and  $(H_{2k}(\mu), \dots)$  in which  $\lambda$  is real and positive and  $\mu$  is either not real or is real and negative, if the number of dashed arrows in (1) is odd;
- (ii)  $(\dots, J_n(0), \dots)$  with  $J_n(0)$  at position  $i \in \{1, \dots, t\}$ ;
- (iii)  $(\dots, A_i, \dots, A_j, \dots)$  in which
  - $(A_i, A_j) = (F_n, G_n)$  or  $(F_n^T, G_n^T)$  if  $\mathcal{A}_i$  and  $\mathcal{A}_j$  have opposite directions in

$$V_1 \xrightarrow{\mathcal{A}_1} V_2 \xrightarrow{\mathcal{A}_2} \dots \xrightarrow{\mathcal{A}_{t-1}} V_t \xrightarrow{\mathcal{A}_t} V_1 \quad (4)$$

- $(A_i, A_j) = (F_n, G_n^T)$  or  $(F_n^T, G_n)$  if  $\mathcal{A}_i$  and  $\mathcal{A}_j$  have the same direction in (4).

Note that  $F_1$  and  $G_1$  in (3) have the size  $0 \times 1$ . It is agreed that there exists exactly one matrix, denoted by  $0_{n0}$ , of size  $n \times 0$  and there exists exactly one matrix, denoted by  $0_{0n}$ , of size  $0 \times n$  for every nonnegative integer  $n$ ; they represent the linear mappings  $0 \rightarrow \mathbb{C}^n$  and  $\mathbb{C}^n \rightarrow 0$  and are considered as zero matrices. Then

$$M_{pq} \oplus 0_{m0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{p0} \\ 0_{mq} & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{mq} \end{bmatrix}$$

and

$$M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \\ 0_{0q} & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \end{bmatrix}$$

for every  $p \times q$  matrix  $M_{pq}$ .

### 3. Proof of Theorem 1

3.1. *Reduction to cycles in which  $t \geq 2$  and  $\mathcal{A}_1, \dots, \mathcal{A}_{t-1}$  are linear*

The matrix of the composition of a linear mapping and a semilinear mapping is given in the following lemma.

**Lemma 2.** *Let  $e_1, e_2, \dots$ , be a basis of a vector space  $U$ ,  $f_1, f_2, \dots$ , be a basis of  $V$ , and  $g_1, g_2, \dots$  be a basis of  $W$ .*

- (a) *The composition of a linear mapping  $\mathcal{A} : U \rightarrow V$  and a semilinear mapping  $\mathcal{B} : V \dashrightarrow W$  is the semilinear mapping with matrix*

$$(\mathcal{B}\mathcal{A})_{ge} = \mathcal{B}_{gf}\mathcal{A}_{fe} \tag{5}$$

- (b) *The composition of a semilinear mapping  $\mathcal{A} : U \dashrightarrow V$  and a linear mapping  $\mathcal{B} : V \rightarrow W$  is the semilinear mapping with matrix*

$$(\mathcal{B}\mathcal{A})_{ge} = \overline{\mathcal{B}_{gf}}\mathcal{A}_{fe} \tag{6}$$

*Proof.* The identity (5) follows from observing that  $\mathcal{A}\mathcal{B}$  is a semilinear mapping and

$$[(\mathcal{B}\mathcal{A})u]_g = [\mathcal{B}(\mathcal{A}u)]_g = \overline{\mathcal{B}_{gf}[\mathcal{A}u]_f} = \overline{(\mathcal{B}_{gf}\mathcal{A}_{fe})[u]_e}$$

for each  $u \in U$ . The identity (6) follows from observing that  $\mathcal{AB}$  is a semilinear mapping and

$$[(\mathcal{BA})u]_g = [\mathcal{B}(\mathcal{A}u)]_g = \mathcal{B}_{gf}[\mathcal{A}u]_f = \mathcal{B}_{gf}\overline{\mathcal{A}_{fe}[u]_e} = \overline{(\overline{\mathcal{B}_{gf}}\mathcal{A}_{fe})[u]_e}$$

for each  $u \in U$ . □

Let  $\mathcal{A}: U \dashrightarrow V$  be a semilinear operator, let  $\mathcal{A}_{fe}$  be its matrix in bases  $e_1, \dots, e_m$  of  $U$  and  $f_1, \dots, f_n$  of  $V$ , and let  $\mathcal{A}_{f'e'}$  be its matrix in other bases  $e'_1, \dots, e'_m$  and  $f'_1, \dots, f'_n$ . Then

$$\mathcal{A}_{f'e'} = \bar{S}_{f \rightarrow f'}^{-1} \mathcal{A}_{fe} S_{e \rightarrow e'}$$

since the right hand matrix satisfies (2) with  $e', f'$  instead of  $e, f$ :

$$\overline{\bar{S}_{f \rightarrow f'}^{-1} \mathcal{A}_{fe} S_{e \rightarrow e'} [v]_{e'}} = S_{f \rightarrow f'}^{-1} \overline{\mathcal{A}_{fe} [v]_e} = S_{f \rightarrow f'}^{-1} [\mathcal{A}v]_f = [\mathcal{A}v]_{f'}$$

In particular, if  $U = V$ , then

$$\mathcal{A}_{e'} = \bar{S}_{e \rightarrow e'}^{-1} \mathcal{A}_e S_{e \rightarrow e'}$$

and so  $\mathcal{A}_{e'}$  and  $\mathcal{A}_e$  are consimilar; recall that two matrices  $A$  and  $B$  are *consimilar* if there exists a nonsingular matrix  $S$  such that  $\bar{S}^{-1}AS = B$ .

The following canonical form of a matrix under consimilarity was obtained in [8, Theorem 3.1]; see also [9, Theorem 4.6.12].

**Theorem 3** ([8, 9]). *Each square complex matrix is consimilar to a direct sum, uniquely determined up to permutation of direct summands, of matrices of the following three types:*

**Type 0:**  $J_k(0)$ ,  $k = 1, 2, \dots$ ;

**Type I:**  $J_k(\lambda)$ ,  $k = 1, 2, \dots$ , in which  $\lambda$  is real and positive;

**Type II:**  $H_{2k}(\mu)$ ,  $k = 1, 2, \dots$ , in which  $\mu$  is either not real or is real and negative.

It suffices to prove Theorem 1 for cycles  $\mathcal{A}$  of the form (1) in which

$$t \geq 2 \text{ and the mappings } \mathcal{A}_1, \dots, \mathcal{A}_{t-1} \text{ are linear} \quad (7)$$

since Theorem 1 for the cycles  $V_1 \xrightarrow{\mathcal{A}_1} V_1$  and  $V_1 \xrightarrow{\mathcal{A}_1} V_1$  of length  $t = 1$  follows from the Jordan canonical form and Theorem 3, and all  $\mathcal{A}_1, \dots, \mathcal{A}_{t-1}$  can be made linear mappings by using the following procedure:

Let not all  $\mathcal{A}_1, \dots, \mathcal{A}_{t-1}$  be linear and let  $\mathcal{A}_{k-1}$  for some  $k \leq t$  be the first semilinear mapping. Denote by  $\mathcal{A}^{(k)}$  the cycle obtained from  $\mathcal{A}$  by replacing the vector space  $V_k$  by the vector space  $\bar{V}_k$  defined as follows:  $\bar{V}_k$  consists of the same elements as  $V_k$ , the addition in  $\bar{V}_k$  is the same as in  $V_k$ , and the multiplication (which we denote by “ $\circ$ ”) in  $\bar{V}_k$  is defined via the multiplication in  $V_k$  by  $c \circ v = \bar{c}v$  for all  $c \in \mathbb{C}$  and  $v \in \bar{V}_k$ . (8)

The semilinear mapping  $\mathcal{A}_{k-1}$  becomes linear in  $\mathcal{A}^{(k)}$ . Indeed, if  $V_{k-1} \xrightarrow{\mathcal{A}_{k-1}} V_k$ , then  $\mathcal{A}_{k-1}(cu) = \bar{c}\mathcal{A}_{k-1}u = c \circ \mathcal{A}_{k-1}u$  for all  $c \in \mathbb{C}$  and  $u \in V_{k-1}$ . If  $V_{k-1} \xleftarrow{\mathcal{A}_{k-1}} V_k$ , then  $\mathcal{A}_{k-1}(c \circ v) = \mathcal{A}_{k-1}(\bar{c}v) = c\mathcal{A}_{k-1}v$  for all  $c \in \mathbb{C}$  and  $v \in V_k$ .

Similarly, the linear or semilinear mapping  $\mathcal{A}_k$  becomes semilinear or, respectively, linear in  $\mathcal{A}^{(k)}$ .

We repeat procedure (8) until obtain a chain in which the first  $t - 1$  mappings are linear.

### 3.2. A canonical form of the system of matrices of $\mathcal{A}_1, \dots, \mathcal{A}_{t-1}$

Deleting  $\mathcal{A}_t$  in (1) satisfying (7), we obtain a system of linear mappings

$$\mathbb{A}: \quad V_1 \xrightarrow{\mathcal{A}_1} V_2 \xrightarrow{\mathcal{A}_2} \dots \xrightarrow{\mathcal{A}_{t-1}} V_t \quad (9)$$

in which each line is either  $\longrightarrow$  or  $\longleftarrow$ . The classification of such systems is well known in the theory of quiver representations (see, for example, Gabriel’s article [5], in which he introduced the notion of quiver representations and described all quivers with a finite number of nonisomorphic indecomposable representations). We recall this classification in Lemma 4 in a form that is used in the next section for reducing the matrix of  $\mathcal{A}_t$  to canonical form.

A *chain of type*  $(p, q)$  of system (9),  $1 \leq p \leq q \leq t$ , is a sequence of nonzero elements

$$(v_p, v_{p+1}, \dots, v_q) \in V_p \times V_{p+1} \times \dots \times V_q$$

such that  $\mathcal{A}_1, \dots, \mathcal{A}_t$  act on them as follows:

$$0 \xrightarrow{\mathcal{A}_1} \dots \xrightarrow{\mathcal{A}_{p-2}} 0 \xrightarrow{\mathcal{A}_{p-1}} v_p \xrightarrow{\mathcal{A}_p} \dots \xrightarrow{\mathcal{A}_{q-1}} v_q \xrightarrow{\mathcal{A}_q} 0 \xrightarrow{\mathcal{A}_{q+1}} \dots \xrightarrow{\mathcal{A}_{t-1}} 0$$

in which the arrows are directed as in (9); for simplicity of notation, we write  $0 \xrightarrow{\mathcal{A}_{p-1}} v_p$  instead of  $0 \xrightarrow{\mathcal{A}_{p-1}} 0$  and  $v_q \xleftarrow{\mathcal{A}_q} 0$  instead of  $0 \xleftarrow{\mathcal{A}_q} 0$ .

**Lemma 4.** (a) *For each system of linear mappings (9), we can choose bases  $E_1, \dots, E_t$  of the spaces  $V_1, \dots, V_t$  so that  $E_1 \cup \dots \cup E_t$  consists of disjoint chains. The sequence of their types*

$$\underbrace{(p_1, q_1), \dots, (p_1, q_1)}_{r_1 \text{ times}}, \dots, \underbrace{(p_s, q_s), \dots, (p_s, q_s)}_{r_s \text{ times}}, \quad (10)$$

*in which  $(p_i, q_i) \neq (p_j, q_j)$  if  $i \neq j$ , is determined by (9) uniquely up to permutation.*

(b) *Suppose that the types in (10) are numbered so that for each two pairs  $(p_k, q_k)$  and  $(p_l, q_l)$*

- (i) *if  $p_k = p_l = 1$  and  $q_k < q_l$ , then either  $V_{q_k} \xrightarrow{\mathcal{A}_{q_k}} V_{q_{k+1}}$  and  $k < l$ , or  $V_{q_k} \xleftarrow{\mathcal{A}_{q_k}} V_{q_{k+1}}$  and  $k > l$ ;*
- (ii) *if  $p_k < p_l$  and  $q_k = q_l = t$ , then either  $V_{p_{k-1}} \xrightarrow{\mathcal{A}_{p_{k-1}}} V_{p_k}$  and  $k > l$ , or  $V_{p_{k-1}} \xleftarrow{\mathcal{A}_{p_{k-1}}} V_{p_k}$  and  $k < l$ .*

*Define nonnegative integers  $m_1, \dots, m_s$  and  $n_1, \dots, n_s$  by (10) as follows:*

$$m_i := \begin{cases} r_i & \text{if } p_i = 1, \\ 0 & \text{if } p_i > 1, \end{cases} \quad n_i := \begin{cases} r_i & \text{if } q_i = t, \\ 0 & \text{if } q_i < t. \end{cases} \quad (11)$$

*Then the following two conditions are equivalent for a pair of nonsingular matrices  $R$  and  $S$ :*

- (i') *There exists another system of bases  $F_1, \dots, F_t$  of the spaces  $V_1, \dots, V_t$  that consists of disjoint chains with the same sequence (10) of their types such that  $R$  and  $S$  are the change of basis matrices from  $E_1$  to  $F_1$  in  $V_1$  and from  $E_t$  to  $F_t$  in  $V_t$ .*
- (ii')  *$R$  and  $S$  are upper block triangular matrices*

$$R = \begin{bmatrix} R_{11} & \cdots & R_{1s} \\ & \ddots & \vdots \\ 0 & & R_{ss} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & \cdots & S_{1s} \\ & \ddots & \vdots \\ 0 & & S_{ss} \end{bmatrix}, \quad (12)$$

in which every  $R_{ii}$  is  $m_i \times n_i$  and every  $S_{ii}$  is  $n_i \times n_i$  (see (11)). If there is  $l$  such that  $(p_l, q_l) = (1, t)$  in (10), then

$$R_{ll} = S_{ll}. \quad (13)$$

*Sketch of the proof.* (a) This statement (in another form) is given in many articles; see, for example [14, Section 4].

(b) (i')  $\Rightarrow$  (ii') Let  $R$  and  $S$  satisfy (i').

The change of basis matrix  $R$  from  $E_1$  to  $F_1$  in  $V_1$  has the upper block triangular form (12) by the following reason. Let  $E_1 \cup \dots \cup E_t$  contain two chains of basis vectors

$$\begin{aligned} \mathbb{E}: \quad & e_1 \text{ --- } \dots \text{ --- } e_{q_k} \xrightarrow{\mathcal{A}_{q_k}} 0 \text{ --- } \dots \text{ --- } 0 \\ \mathbb{E}': \quad & e'_1 \text{ --- } \dots \text{ --- } e'_{q_k} \xrightarrow{\mathcal{A}_{q_k}} e'_{q_k+1} \text{ --- } \dots \text{ --- } e'_{q_l} \text{ --- } 0 \text{ --- } \dots \text{ --- } 0 \end{aligned}$$

of types  $(1, q_k)$  and  $(1, q_l)$  with  $q_k < q_l$ . We consider two cases according to the direction of  $\mathcal{A}_{q_k}$ :

- If  $V_{q_k} \xrightarrow{\mathcal{A}_{q_k}} V_{q_k+1}$ , then

$$\mathbb{E}' + \alpha \mathbb{E}: \quad e'_1 + \alpha e_1 \text{ --- } \dots \text{ --- } e'_{q_k} + \alpha e_{q_k} \xrightarrow{\mathcal{A}_{q_k}} e'_{q_k+1} \text{ --- } \dots \text{ --- } 0$$

for each  $\alpha \in \mathbb{C}$ , and so we can replace the chain  $\mathbb{E}'$  by  $\mathbb{E}' + \alpha \mathbb{E}$  in  $E_1 \cup \dots \cup E_t$ . Thus, the basis vector  $e'_1 \in E_1$  can be replaced by  $e'_1 + \alpha e_1$ , and so the block  $R_{kl}$  in (12) is arbitrary. Due to the condition (i) in (b),  $k < l$ .

- If  $V_{q_k} \xleftarrow{\mathcal{A}_{q_k}} V_{q_k+1}$ , then

$$\mathbb{E} + \alpha \tilde{\mathbb{E}}': \quad e_1 + \alpha e'_1 \text{ --- } \dots \text{ --- } e_{q_k} + \alpha e'_{q_k} \xleftarrow{\mathcal{A}_{q_k}} 0 \text{ --- } \dots \text{ --- } 0,$$

in which  $\alpha \in \mathbb{C}$  and  $\tilde{\mathbb{E}}' = \{e'_1, \dots, e'_{q_k}\}$  is the subchain of  $\mathbb{E}'$ . Hence we can replace the chain  $\mathbb{E}$  by  $\mathbb{E} + \alpha \tilde{\mathbb{E}}'$ , and so  $e_1 \in E_1$  can be replaced by  $e_1 + \alpha e'_1$ . Therefore, the block  $R_{lk}$  in (12) is arbitrary. Due to the condition (i) in (b),  $k > l$ .



Analogously, the change of basis matrix  $S$  from  $E_t$  to  $F_t$  in  $V_t$  has the upper block triangular form (12) by the following reason. Let  $E_1 \cup \dots \cup E_t$  contain two chains of basis vectors

$$\begin{aligned}\mathbb{E}: & 0 \cdots 0 \xrightarrow{\mathcal{A}_{p_k-1}} e_{p_k} \cdots e_t \\ \mathbb{E}': & 0 \cdots 0 \xrightarrow{\mathcal{A}_{p_k-1}} e'_{p_k} \cdots e'_t\end{aligned}$$

of types  $(p_k, t)$  and  $(p_l, t)$  with  $p_l < p_k$ .

- If  $V_{p_k-1} \xrightarrow{\mathcal{A}_{p_k-1}} V_{p_k}$ , then

$$\mathbb{E} + \alpha \tilde{\mathbb{E}}': 0 \cdots 0 \xrightarrow{\mathcal{A}_{p_k-1}} e_{p_k} + \alpha e'_{p_k} \cdots e_t + \alpha e'_t,$$

in which  $\alpha \in \mathbb{C}$  and  $\tilde{\mathbb{E}}' = \{e'_{p_k}, \dots, e'_t\}$  is the subchain of  $\mathbb{E}'$ . Hence, we can replace the chain  $\mathbb{E}$  by  $\mathbb{E} + \alpha \tilde{\mathbb{E}}'$ , and so the basis vector  $e_t \in E_t$  can be replaced by  $e_t + \alpha e'_t$ . Therefore, the block  $R_{lk}$  in (12) is arbitrary. Due to the condition (ii) in (b),  $k > l$ .

- If  $V_{p_k-1} \xleftarrow{\mathcal{A}_{p_k-1}} V_{p_k}$ , then

$$\mathbb{E}' + \alpha \mathbb{E}: 0 \cdots e'_{p_k-1} \xleftarrow{\mathcal{A}_{p_k-1}} e'_{p_k} + \alpha e_{p_k} \cdots e'_t + \alpha e_t$$

for each  $\alpha \in \mathbb{C}$ , and so we can replace the chain  $\mathbb{E}'$  by  $\mathbb{E}' + \alpha \mathbb{E}$ . Thus, the basis vector  $e'_t \in E_t$  can be replaced by  $e'_t + \alpha e_t$ . Therefore, the block  $R_{kl}$  in (12) is arbitrary. Due to the condition (ii) in (b),  $k < l$ .

Let  $l$  be such that  $(p_l, q_l) = (1, t)$  in (10), let  $r_l \geq 2$ , and let

$$\begin{aligned}\mathbb{E}: & e_1 \cdots e_t \\ \mathbb{E}': & e'_1 \cdots e'_t\end{aligned}$$

be two chains. Then

$$\mathbb{E} + \alpha \mathbb{E}': e_1 + \alpha e'_1 \cdots e_t + \alpha e'_t$$

for each  $\alpha \in \mathbb{C}$ . Hence the basis vectors  $e_1 \in E_1$  and  $e_t \in E_t$  can be simultaneously replaced by  $e_1 + \alpha e'_1$  and  $e_t + \alpha e'_t$ . Therefore,  $R_{ll} = S_{ll}$  in (12).

We have shown that  $R$  and  $S$  satisfy (ii').

(i')  $\Leftarrow$  (ii') This implication follows from the above reasoning.  $\square$

### 3.3. A canonical form of the matrix of $\mathcal{A}_t$

Let  $\mathcal{A}$  be a cycle of linear and semilinear mappings (1) satisfying the condition (7). We suppose that  $V_1 \xrightarrow{\mathcal{A}_t} V_t$  or  $V_1 \xrightarrow{-\mathcal{A}_t} V_t$  (if  $V_1 \xleftarrow{\mathcal{A}_t} V_t$  or  $V_1 \xleftarrow{-\mathcal{A}_t} V_t$ , then we renumber the vector spaces  $V_1, \dots, V_t$  in the reverse order).

Deleting  $\mathcal{A}_t$  from (1), we obtain the chain of linear mappings (9). By Lemma 4, there exists a system of bases  $E_1, \dots, E_t$  of the spaces  $V_1, \dots, V_t$  that consists of disjoint chains. Moreover, the bases  $E_1$  and  $E_t$  are determined up to replacement by bases  $F_1$  and  $F_t$  such that the change of basis matrices  $R$  and  $S$  have the form (12). Thus, the matrix  $A_t$  of the linear or semilinear mapping  $\mathcal{A}_t$  is reduced by transformations

$$A_t \mapsto S^{-1}A_tR \text{ or } \bar{S}^{-1}A_tR, \quad R \text{ and } S \text{ have the form (12)}. \quad (14)$$

This leads to the following definition.

**Definition 5.** By a *marked block matrix* we mean a block matrix in which some of square blocks are crossed along their main diagonal by a full or dashed line and each horizontal or vertical strip contains at most one crossed block. A *block matrix problem* is the canonical form problem for marked block matrices with respect to the following admissible transformations:

- (i) arbitrary elementary transformations within vertical and horizontal strips such that each crossed block is transformed by similarity transformations if its cross-line is full and by consimilarity transformations if the cross-line is dashed;
- (ii) additions of a row multiplied by scalar to a row of a horizontal strip that is located above, and of a column multiplied by scalar to a column of a vertical strip that is located to the right.

We say that  $M$  and  $N$  are *equivalent* if  $M$  is reduced to  $N$  by transformations (i) and (ii). The problem is to find a canonical form of a marked block matrix up to equivalence.

We use the methods of [13], in which the block matrix problem was solved for marked block matrices without blocks that are crossed by dashed lines.

It follows from the form of  $R$  and  $S$  in (12) that the canonical form problem for  $A_t$  with respect to transformations (14) is the block matrix problem for  $A_t$  partitioned into horizontal strips conformally to the partition of  $S$

and into vertical strips conformally to the partition of  $R$ ; the condition (13) means that the  $(l, l)$  block of  $A_t$  is crossed by a full or dash line. (Note that  $A_t$  may contain horizontal strips with no rows and vertical strips with no columns, which follows from (11).)

Let  $M = [M_{ij}]$  and  $N = [N_{ij}]$  be two block matrices with the same number of horizontal strips, the same number of vertical strips, the same disposition of blocks crossed by full lines, and the same disposition of blocks crossed by dashed lines. The *block direct sum* of  $M$  and  $N$  is the block matrix

$$M \boxplus N := [M_{ij} \oplus N_{ij}]$$

with the same disposition of blocks crossed by full lines and the same disposition of blocks crossed by dashed lines. We say that  $M$  is *indecomposable* if it is not 0-by-0 and it is not equivalent to a block direct sum of matrices of smaller sizes.

We say that a matrix is *empty* if it does not contain rows or columns.

**Lemma 6.** *Let a marked block matrix  $M$  be indecomposable. Then either*

- (a) *all blocks of  $M$  are empty except for one crossed block that is nonsingular, or*
- (b)  *$M$  is equivalent to a matrix whose entries are only 0's and 1's with at most one 1 in each row and each column.*

*Proof.* Let  $M \neq 0$ . We use induction on the size of  $M$ . Let  $M_{pq}$  be the lowest nonzero block in the first nonzero vertical strip.

*Case 1:*  $M_{pq}$  is not crossed. We reduce it by transformations (i) to the form

$$M_{pq} = \begin{bmatrix} 0 & I_k \\ 0 & 0 \end{bmatrix}, \quad k \geq 1, \quad (15)$$

and then use only those transformations (i) and (ii) that preserve it.

The matrix (15) is partitioned into 2 horizontal and 2 vertical strips. We extend this partition to the entire  $p$ th horizontal strip and to the entire  $q$ th vertical strip of  $M$ . Using transformations (ii), we make zero all entries over  $I_k$  and to the right of  $I_k$  in the new strips.

Let the new division lines do not pass through crossed blocks. Then  $I_k$  is a direct summand of  $M$ . Since  $M$  is indecomposable, we have that  $k = 1$ ,  $M_{pq} = I_1$ , and all other blocks are empty, which proves the lemma in this case.

Let a new division line pass through a crossed block  $M_{lr}$  ( $l = p$  or  $r = q$ ). Then we draw the perpendicular division line such that  $M_{lr}$  is partitioned into 4 subblocks with square diagonal blocks (they are crossed by the full or dash line that crosses  $M_{lr}$ ):

$$M_{lr} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \begin{array}{l} A \text{ and } D \text{ are crossed,} \\ A \text{ is } k \times k \text{ if } l = p, D \text{ is } k \times k \text{ if } r = q. \end{array} \quad (16)$$

If a new division line passes through another crossed block, we repeat this procedure.

Denote by  $M'$  the marked block matrix obtained from  $M$  by deleting the new vertical and horizontal strips that pass through  $I_k$  in  $M_{pq}$  (if  $M$  is  $m$ -by- $n$ , then  $M'$  is  $(m - k)$ -by- $(n - k)$ ). It is easy to see that all transformations (i) and (ii) with  $M'$  can be obtained from transformations (i) and (ii) with  $M$ . Reasoning by induction, we assume that the lemma holds for  $M'$ .

Note that  $M'$  cannot satisfy the condition (a) since  $M'$  contains a strip with  $k$  rows or columns and without crossed blocks; this strip is obtained from the strip of  $M$  that contains  $A$  in (16) if  $l = p$  or  $D$  if  $r = q$ . Therefore,  $M'$  satisfies the condition (b), and so  $M$  satisfies (b) too.

*Case 2:  $M_{pq}$  is crossed.* The Jordan canonical form for similarity (if the cross-line is full) and Theorem 3 (if the cross-line is dashed) ensure that  $M_{pq}$  can be reduced to the form  $N \oplus J$ , in which  $N$  is nonsingular and  $J$  is a nilpotent Jordan matrix.

If  $N$  is nonempty, then we make zero all entries over  $N$  and to the right of  $N$  by transformations (ii). Since  $M$  is indecomposable,  $M_{pq} = N$  and the other blocks of  $M$  are empty, and so  $M$  satisfies the condition (a). Thus, we can suppose that  $M = J$ .

Collecting the Jordan blocks of the same size by permutations of rows and the same permutations of columns, we reduce  $M_{pq}$  to the form

$$M_{pq} = \begin{array}{c} \begin{array}{|c|c|c|c|c|c|c|} \hline 0_u & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & I_v & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & I_w & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 0 & I_w & \dots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \hline \end{array} & \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \vdots \end{array} \end{array}, \quad u, v, w, \dots \geq 0. \quad (17)$$

We extend the obtained partition of  $M_{pq}$  to the entire  $p$ th horizontal strip and the entire  $q$ th vertical strip of  $M$  and make zero all entries over and to the right of  $I_v, I_w, I_w, \dots$

Denote by  $M'$  the submatrix of  $M$  obtained from  $M$  by deleting the “new” horizontal and vertical strips containing  $I_v, I_w, I_w, \dots$ . In particular,  $M_{pq}$  converts to

$$M'_{pq} = \begin{array}{c|ccc|c} \hline 0_u & 0 & 0 & \cdots & 1 \\ \hline 0 & 0_v & 0 & \cdots & 3 \\ \hline 0 & 0 & 0_w & \cdots & 6 \\ \hline \vdots & \vdots & \vdots & \ddots & \\ \hline \end{array}$$

1      2      4

The set of crossed blocks of  $M'$  consists of the crossed blocks of  $M$  (except for  $M_{pq}$ ) and the diagonal blocks  $0_u, 0_v, 0_w, \dots$  of  $M'_{pq}$ . If  $M_{pq}$  is crossed by a full line, then  $0_u, 0_v, 0_w, \dots$  are crossed by full lines too. If  $M_{pq}$  is crossed by a dash line, then all odd-numbered diagonal blocks  $0_u, 0_w, \dots$  of  $M_{pq}$  (they are obtained from the Jordan blocks of  $M_{pq}$  of odd sizes) are crossed by dash lines; the even-numbered diagonal blocks  $0_v, \dots$  of  $M_{pq}$  are crossed by full lines.

Denote by  $M''$  the marked block matrix obtained from  $M'$  by arrangement of its “new” vertical strips (passing through  $M'_{pq}$ ) in reverse order. In particular,  $M'_{pq}$  converts to

$$M''_{pq} = \begin{array}{c|ccc|c} \hline \cdots & 0 & 0 & 0_u & 1 \\ \hline \cdots & 0 & 0_v & 0 & 3 \\ \hline \cdots & 0_w & 0 & 0 & 6 \\ \hline \vdots & \vdots & \vdots & \vdots & \\ \hline \end{array}$$

4      2      1

Let us prove that

if  $M$  is reduced by transformations (i) and (ii) that preserve  $M_{pq}$  (which is of the form (17)), then  $M''$  is reduced by transformations (i) and (ii). (18)

We first prove (18) for transformations (ii). Preserving  $M_{pq}$ , we can make the following transformations with strips of  $M$  that  $M_{pq}$ :

- Add columns of the 2nd vertical strip to columns of the 1st vertical strip. Since  $M_{pq}$  is transformed by (con)similarity transformations, we

must do the (con)inverse transformations with rows of  $M_{pq}$ : to subtract the corresponding columns (or the complex conjugate columns) of the 1st horizontal strip from the 2nd horizontal strip, which does not change  $M_{pq}$  since its 1st horizontal strip is zero. Thus, we can add in  $M''_{pq}$  columns passing through  $0_v$  to columns passing through  $0_u$ .

- Add columns of the 4th vertical strip to columns of the 1st vertical strip; the (con)inverse transformations does not change  $M_{pq}$ . Thus, we can add in  $M''_{pq}$  columns passing through  $0_w$  to columns passing through  $0_u$ .
- Add columns of the 4th vertical strip to columns of the 2nd vertical strip. The (con)inverse transformations with rows of  $M_{pq}$  spoil the (4,3) block. It is restored by additions of columns of the 5th vertical strip; the (con)inverse transformations do not change  $M_{pq}$ . Thus, we can add in  $M''_{pq}$  columns passing through  $0_w$  to columns passing through  $0_v$ .  
...
- Add rows of the 3rd and 6th horizontal strips to rows of the 1st horizontal strip; the (con)inverse transformations with columns do not change  $M_{pq}$ . Thus, we can add in  $M''_{pq}$  rows passing through  $0_v$  and  $0_w$  to rows passing through  $0_u$ .
- Add rows of the 6th horizontal strip to rows of the 3rd horizontal strip. The (con)inverse transformations with columns of  $M_{pq}$  spoil the (2,6) block. It is restored by additions of rows of the 5th horizontal strip; the (con)inverse transformations do not change  $M_{pq}$ . Thus, we can add in  $M''_{pq}$  rows passing through  $0_w$  to rows passing through  $0_v$ .  
...

Therefore, preserving  $M_{pq}$  we can make transformations (ii) with  $M''$ , which proves (18) for transformations (ii).

Let us prove (18) for transformations (i).

If  $M_{pq}$  is crossed by a full line (i.e.,  $M_{pq}$  is transformed by similarity transformations), then we take  $S := S_1 \oplus S_2 \oplus S_2 \oplus S_3 \oplus S_3 \oplus S_3 \oplus \dots$  (in which  $S_1$  is  $u \times u$ ,  $S_2$  is  $v \times v, \dots$ ) and obtain  $S^{-1}M_{pq}S = M_{pq}$ . Thus,  $M'_{pq}$  can be reduced by transformations

$$(S_1 \oplus S_2 \oplus S_3 \oplus \dots)^{-1} M'_{pq} (S_1 \oplus S_2 \oplus S_3 \oplus \dots),$$

and so  $M''_{pq}$  can be reduced by transformations (i).

Assume now that  $M_{pq}$  is crossed by a dashed line; that is,  $M_{pq}$  is transformed by consimilarity transformations. Then

$$\bar{S}^{-1}M_{pq}S = M_{pq} \quad \text{if } S := S_1 \oplus S_2 \oplus \bar{S}_2 \oplus S_3 \oplus \bar{S}_3 \oplus S_3 \oplus \dots$$

which is illustrated as follows:

	$S_1$	$S_2$	$\bar{S}_2$	$S_3$	$\bar{S}_3$	$S_3$	
$\bar{S}_1^{-1}$	$0_u$	0	0	0	0	0	...
$\bar{S}_2^{-1}$	0	0	$I_v$	0	0	0	...
$S_2^{-1}$	0	$0_v$	0	0	0	0	...
$\bar{S}_3^{-1}$	0	0	0	0	$I_w$	0	...
$S_3^{-1}$	0	0	0	0	0	$I_w$	...
$\bar{S}_3^{-1}$	0	0	0	$0_w$	0	0	...
	...	...	...	...	...	...	$\ddots$

Thus,  $0_u, 0_w, \dots$  are transformed by consimilarity transformations and  $0_v, \dots$  by similarity transformations. This proves (18).

Reasoning by induction, we assume that the lemma holds for  $M''$ . The matrix  $M''$  cannot satisfy the condition (a) of the lemma since  $M''_{pq}$  is nonempty. Hence,  $M''$  satisfies (b), then  $M''$  satisfies (b) too.  $\square$

### 3.4. Completion of the proof of Theorem 1

Let  $\mathcal{A}$  be a system of linear and semilinear mappings (1) whose sequence of matrices of  $\mathcal{A}_1, \dots, \mathcal{A}_t$  cannot be decomposed into a direct sum of systems of matrices of smaller sizes. By Lemma 4, there exists a system of bases  $E_1, \dots, E_t$  of the spaces  $V_1, \dots, V_t$  that falls into disjoint chains. By Lemma 6, the bases  $E_1, \dots, E_t$  can be chosen such that either

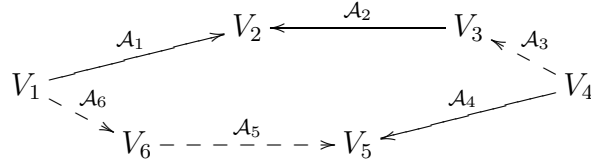
- (a) the matrices  $A_1, \dots, A_{t-1}$  of  $\mathcal{A}_1, \dots, \mathcal{A}_{t-1}$  are the identity and the matrix  $A_t$  of  $\mathcal{A}_t$  is nonsingular, or
- (b) if  $\mathcal{A}_i : V_k \rightarrow V_l$  or  $\mathcal{A}_i : V_k \dashrightarrow V_l$  and  $e \in E_k$ , then  $\mathcal{A}_i e \in E_l$  or  $\mathcal{A}_i e = 0$ ; moreover, if  $e, f \in E_k$  and  $\mathcal{A}_i e = \mathcal{A}_i f \neq 0$ , then  $e = f$ .

Consider the case (a). Let  $S_1, \dots, S_t$  be the change of basis matrices that preserve  $A_1 = \dots = A_{t-1} = I$ , then  $S_i = S_{i+1}$  if the arrow between  $i$  and  $i+1$  is full and  $S_i = \bar{S}_{i+1}$  if the arrow is dashed. Therefore,  $A_t$  is reduced by similarity transformations  $S_1^{-1}A_tS_1$  or consimilarity transformations  $\bar{S}_1^{-1}A_tS_1$

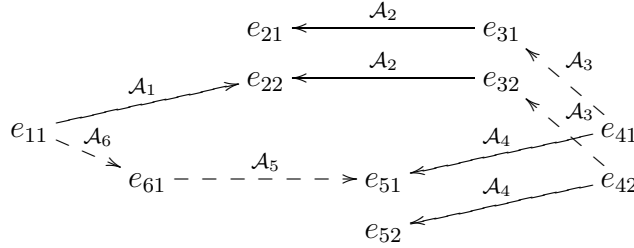
if the number of dashed arrows in (1) is even or odd, respectively. Using the Jordan canonical form or Theorem 3, we obtain a sequence of matrices (i) from Theorem 1.

Consider the case (b). Let us construct the directed graph, whose set of vertices is the set of basis vectors  $E_1 \cup \dots \cup E_t$  and there is an arrow from  $u$  to  $v$  if and only if  $A_i u = v$  for some  $i = 1, \dots, t$ . In the case (b), this graph is a disjoint union of chains. Since the sequence of matrices of  $\mathcal{A}_1, \dots, \mathcal{A}_t$  cannot be decomposed into a direct sum, the graph is connected, and so it is a chain.

For example, a system



may have the chain



Its mappings  $\mathcal{A}_1, \dots, \mathcal{A}_6$  are given by the matrices

$$A_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_2 = A_3 = A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_6 = [1].$$

They form a sequence (iii) from Theorem 1 with  $(A_i, A_j) = (A_1, A_5) = (F_2^T, G_2^T)$ . It is easy to see that all matrix sequences (ii) and (iii) from Theorem 1 can be obtained analogously.

We have proved that for each cycle of linear and semilinear mappings (1) there exist bases of the spaces  $V_1, \dots, V_t$  in which the sequence of matrices of  $\mathcal{A}_1, \dots, \mathcal{A}_t$  is a direct sum of sequences of the form (i)–(iii); the uniqueness of this direct sum follows from the Krull–Schmidt theorem for additive categories [1, Chapter I, Theorem 3.6] (it holds for cycles (1) since they form an additive category in which all idempotents split).



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